

Leading quantum correction to energy of “short” spiky strings

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ABSTRACT: We consider semiclassical quantization of spiky strings spinning in AdS_3 part of $AdS_5 \times S^5$ using integrability-based (algebraic curve) method. In the “short string” (small spin) limit the expansion of string energy starts with its flat-space expression. We compute the leading quantum string correction to “short” spiky string energy and find the explicit form of the corresponding 1-loop coefficient a_{01} . It turns out to be rational and expressed in terms of the harmonic sums as functions of the number n of spikes. In the special case of $n = 2$ when the spiky string reduces to the single-folded spinning string the coefficient a_{01} takes the value $(-1/4)$ found in arXiv:1102.1040. We also consider a similar computation for the m -folded string and more general spiky string with an extra “winding” number, finding similar expressions for a_{01} . These results may be useful for a description of energies of higher excited states in the quantum $AdS_5 \times S^5$ string spectrum, generalizing earlier discussions of the string counterparts of the Konishi operator.

KEYWORDS: AdS/CFT spectrum, spiky strings, algebraic curve approach

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1 Introduction and summary

The simplest example of the AdS/CFT duality [1] states the equivalence between the spectrum of the planar $\mathcal{N} = 4$ supersymmetric gauge theory and the spectrum of free closed quantum superstring propagating in $AdS_5 \times S^5$. The gauge-theory spectrum can be described in two equivalent ways: either as a list of possible energies of states on $\mathbb{R} \times S^3$ (as functions of various quantum numbers and 't Hooft coupling λ) or as a list of anomalous dimensions of conformal primary operators on $\mathbb{R}^{1,3}$ (determined by diagonalization of anomalous dimension matrix for single-trace gauge-invariant operators). Similarly, the string spectrum is given by the AdS_5 energies E of string states on a cylinder $\mathbb{R} \times S^1$ (in, *e.g.*, a light-cone gauge approach) or can be found from the marginality condition for the corresponding string vertex operators on a plane $\mathbb{R}^{1,1}$ (by diagonalizing the 2-d anomalous dimension matrix).

The states can be labeled by the conserved charges C , i.e. by the five spins $(S_{1,2}, J_{1,2,3})$ corresponding to the bosonic subgroup $SO(2, 4) \times SO(6)$ of the $PSU(2, 2|4)$ symmetry group as well as by higher hidden charges. The AdS/CFT correspondence then implies that

$$E_{\text{gauge}}(\lambda, C) = E_{\text{string}}(\sqrt{\lambda}, C) , \quad (1.1)$$

where $\frac{\sqrt{\lambda}}{2\pi} = \frac{R^2}{2\pi\alpha'}$ is the $AdS_5 \times S^5$ string tension.

In the strong-coupling ($\lambda \gg 1$) expansion, one expects that massive quantum string states should probe a near-flat region of $AdS_5 \times S^5$ and thus should have $E \sim \sqrt[4]{\lambda}$ [2]. More generally, considerations based on solving the 2d marginality condition [3] perturbatively in $\frac{1}{\sqrt{\lambda}} \ll 1$ for fixed charges suggest [4] that (up to a possible shift of E by a constant)

$$E^2 = 2N\sqrt{\lambda} + b_0 + \frac{b_1}{\sqrt{\lambda}} + \frac{b_2}{(\sqrt{\lambda})^2} + \dots , \quad (1.2)$$

where N is the flat-space level number. As was argued in [4–6], one can attempt to find quantum string energies by starting with the semiclassical strings with fixed parameters $\mathcal{C} = \frac{C}{\sqrt{\lambda}}$ and then take the “short” string limit $\mathcal{C} \rightarrow 0$. Indeed, for quantum strings with fixed charges C the limit $\sqrt{\lambda} \gg 1$ implies $\mathcal{C} = \frac{C}{\sqrt{\lambda}} \rightarrow 0$. Assuming commutativity of the limits, that suggests a possibility to compute the subleading terms in the above expansion by using the semiclassical string theory methods. The semiclassical string expansion gives¹

$$E = \sqrt{\lambda} \mathcal{E}_0(\mathcal{C}) + \mathcal{E}_1(\mathcal{C}) + \frac{1}{\sqrt{\lambda}} \mathcal{E}_2(\mathcal{C}) + \dots . \quad (1.3)$$

Replacing \mathcal{C} by $\frac{C}{\sqrt{\lambda}}$ and re-expanding in large λ for fixed C one should find that E takes the form consistent with (1.2):

$$E = \sqrt[4]{\lambda} \left(k_1 + \frac{k_2}{\sqrt{\lambda}} + \frac{k_3}{(\sqrt{\lambda})^2} + \dots \right) . \quad (1.4)$$

This “semiclassical” approach was successfully applied to the case of the “short” string states representing members of the Konishi multiplet [4, 6–9], matching the results of the weak-coupling TBA approach extrapolated to strong coupling [7, 10–12].

It is of obvious interest to extend this approach to other quantum string states. The semiclassical analog of the Konishi representative $\text{Tr}(Z D^2 Z)$ in the rank-1 $\mathfrak{sl}(2)$ sector is the “short” folded spinning string [4, 6, 7]. Beyond this twist 2 state, the $\mathfrak{sl}(2)$ sector contains, in particular, the operators of the following schematic form

$$\mathcal{O} = \text{Tr}(D^{s_1} Z \dots D^{s_J} Z) , \quad (1.5)$$

where $S = \sum_{i=1}^J s_i$ is the total spin, J is the R-charge, and D are light-cone projected covariant derivatives. Highest weight states of this form are expected [13–18] to be dual

¹Here we assume that the 1-loop string correction does not contain “non-analytic” terms which cannot appear in the vertex operator (2d anomalous dimension) approach [4]. This will be indeed so in the cases discussed below treated in the algebraic curve approach.

to various spinning string solutions, like the symmetric “spiky” string in AdS_3 found in [19] or its generalization to $AdS_3 \times S^1$ discussed in [20].²

In the weak-coupling gauge theory the operators (1.5) correspond to eigenstates of the $\mathfrak{sl}(2)$ spin chain of length J . The large S limit is a semiclassical limit [13] with $1/S \sim \hbar$. The limit when all s_i in (1.5) are large was analyzed in detail in [17, 18, 23]. The corresponding Bethe ansatz solutions are associated with sectors characterized by a positive integer $n \leq J$. For each n , their description involves a genus $n - 2$ algebraic curve Γ_n with $n - 2$ moduli q_3, \dots, q_n which are higher conserved charges of the spin chain. They are quantized according to suitable Bohr-Sommerfeld conditions making the spectrum discrete.

On the semiclassical string theory side, it is possible to show that the spiky strings are finite gap solutions also associated with a spectral curve that can be put into one-to-one correspondence with the gauge theory curve [17, 18]. Besides, it is possible to determine the cut structure in few special cases. The all-order Bethe ansatz description of the spiky string solutions was presented in [24, 25].

Below we will be interested in two particular solutions. The first one is the “*symmetric spiky string*” solution spinning in AdS_3 and having also an angular momentum J in S^5 . This is a configuration with n spikes symmetrically distributed around the AdS_3 center. The second solution is the *m-folded string*, i.e. a closed string folded on itself m times with spin in AdS_3 and orbital momentum in S^5 [26, 27]. In flat space ($ds^2 = -dt^2 + dx^2 + dy^2$) these solutions are given by (in conformal gauge)

$$\text{spiky : } \begin{cases} t = A(n-1)\tau, & \sigma_{\pm} \equiv \tau \pm \sigma \\ x = \frac{1}{2}A \left(\cos[(n-1)\sigma_+] + (n-1) \cos \sigma_- \right), \\ y = \frac{1}{2}A \left(\sin[(n-1)\sigma_+] + (n-1) \sin \sigma_- \right), \\ E = \sqrt{\frac{4(n-1)}{n\alpha'}} S, & S = \frac{n(n-1)}{4\alpha'} A^2 \end{cases} \quad (1.6)$$

$$\text{m-folded : } \begin{cases} t = A m \tau, \\ x = A \sin(m\sigma) \cos(m\tau) = \frac{1}{2}A \left(\sin(m\sigma_+) + \sin(m\sigma_-) \right), \\ y = A \sin(m\sigma) \sin(m\tau) = \frac{1}{2}A \left(-\cos(m\sigma_+) + \cos(m\sigma_-) \right), \\ E = \sqrt{\frac{2m}{\alpha'}} S, & S = \frac{m}{2\alpha'} A^2. \end{cases} \quad (1.7)$$

One can compute the explicit values of the charges q_j for the corresponding two $AdS_5 \times S^5$ solutions as was shown in [18]. The result is particularly simple since most of the cuts collapse and we are left with a 2-cut solution, asymmetric in the spiky string case. This fact is a major technical simplification and it also shows that these two solutions can be viewed as direct generalizations of the single-folded spinning string (which for $S = J = 2$ is dual to the Konishi state in the $\mathfrak{sl}(2)$ sector).

It is then natural to try to compute the leading quantum string corrections to the corresponding “*short*” string energies. Instead of attempting the direct semiclassical quantization of the superstring, one may utilize the integrability by applying the algebraic curve

²In the Pohlmeyer-reduced description these configurations can be viewed as multi-soliton solutions of a generalized sinh-Gordon model [21, 22].

approach developed, in particular, in [28–30]. This would be a direct generalization of the 1-folded string computation presented in [7].

This is the problem that we address in the main part of this paper. Our results may be summarized as follows. The short string expansion of the energy of the symmetric spiky string with n spikes, spin S and angular momentum J has the form

$$E_{\text{spiky}} = \sqrt{4\left(1 - \frac{1}{n}\right) \sqrt{\lambda} S} \left[1 + \frac{1}{\sqrt{\lambda}} \left(\underbrace{\frac{2n^2 - 5n + 5}{4n(n-1)} S + \frac{n}{8(n-1)} \frac{J^2}{S}}_{\text{classical}} + \underbrace{a_{01}^{\text{spiky}}(n)}_{\text{quantum}} \right) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right], \quad (1.8)$$

where the “classical” terms come from the small spin expansion of the classical spiky string energy [19] while $a_{01}^{\text{spiky}}(n)$ encodes our result for the leading one-loop string correction. The case of the folded string is $n = 2$. The corresponding expression for the square of the energy has a more transparent structure consistent with (1.2) (as explained above, here we assume that S, J, n are fixed while $\lambda \gg 1$)

$$E_{\text{spiky}}^2 = 4\left(1 - \frac{1}{n}\right) \sqrt{\lambda} S \left[1 + \frac{2}{\sqrt{\lambda}} a_{01}^{\text{spiky}}(n) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right] + J^2 + 4\left(1 - \frac{5}{2n} + \frac{5}{2n^2}\right) S^2 + \dots \quad (1.9)$$

The similar expansion of the energy of the m -folded string reads

$$E_{\text{folded}} = \sqrt{2m \sqrt{\lambda} S} \left[1 + \frac{1}{\sqrt{\lambda}} \left(\underbrace{\frac{3}{8m} S + \frac{J^2}{4mS}}_{\text{classical}} + \underbrace{a_{01}^{\text{folded}}(m)}_{\text{quantum}} \right) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right], \quad (1.10)$$

leading to

$$E_{\text{folded}}^2 = 2m \sqrt{\lambda} S \left[1 + \frac{2}{\sqrt{\lambda}} a_{01}^{\text{folded}}(m) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right] + J^2 + \frac{3}{2} S^2 + \dots \quad (1.11)$$

The standard folded string case is $m = 1$, i.e.

$$E_{\text{spiky}}(n = 2) = E_{\text{folded}}(m = 1). \quad (1.12)$$

Our main results are the closed expressions for the one-loop contributions $a_{01}^{\text{spiky}}(n)$ and $a_{01}^{\text{folded}}(m)$ ³:

$$a_{01}^{\text{spiky}}(n) = -\frac{1}{8} + \frac{1}{2} q(n-1), \quad (1.13)$$

$$a_{01}^{\text{folded}}(m) = q(m), \quad (1.14)$$

³Note that the leading quantum correction a_{01} is thus rational. This short string correction should be captured by the asymptotic Bethe Ansatz equations (“wrapping” contributions should start to appear at $\mathcal{O}(S^2)$ order, cf. [31]). Analogous coefficients appearing at higher loop orders are expected to be rational combinations of ζ -numbers with increasing transcendentality. An example is the ζ_3 term in the $\mathcal{O}(\lambda^{-3/4})$ coefficient for the energy of the Konishi operator [9]. For the spiky and m -folded string, these coefficients will be generically dependent on the number of spikes and m .

where the function $q(r)$ is given by the following combination of the harmonic sums⁴

$$q(r) = -\frac{3}{4r} + 2H_r - H_{2r} , \quad (1.15)$$

$$H_r \equiv \sum_{\ell=1}^r \frac{1}{\ell} , \quad H_0 = 0 . \quad (1.16)$$

In the special case of $n = 2$ or $m = 1$ corresponding to the 1-folded spinning string we find, in agreement with [7]

$$a_{01}^{\text{spiky}}(2) = a_{01}^{\text{folded}}(1) = q(1) = -\frac{1}{4} . \quad (1.17)$$

Note that for positive integers $m, n > 1$ one has $2m > 4(1 - \frac{1}{n})$ so that the spiky string with same spin S as the multifolded string with $m > 1$ has lower energy – it corresponds to a state on a lower flat space string level (the single-folded string has, of course lower energy than the spiky string with $n > 2$).

The matching of “short” string energies and anomalous dimensions of finite-length gauge theory operators requires a precise 1-1 mapping of the string states and the operators. This is non-trivial as remarked in [17]. Indeed, the bound $n \leq J$ holds in the perturbative gauge theory, but does not appear to be present on the semiclassical string theory side. Recently, the mirror TBA equations of [32] were solved numerically starting at weak coupling for several two-particle states dual to $\mathcal{N} = 4$ SYM operators from the $\mathfrak{sl}(2)$ sector with various values of the charge J and mode number n [12]. The interpolation of the results to strong coupling for low values of n does not appear to agree with the explicit values of our re-expansion (1.4). This is not totally surprising since moderate values of n at fixed J fall in the *middle* of the allowed range $n \leq J$ whereas on the semiclassical string theory side n is unbounded.⁵

Nevertheless, it is of interest to try to apply (1.9) to a simple generalization of the Konishi operator ($S = J = 2$) in the $\mathfrak{sl}(2)$ sector. This is a particular highest weight state with $S = J = 3$ that can be schematically represented by an operator

$$\mathcal{O}_3 = \text{Tr}(DZDZDZ) + \dots , \quad (1.18)$$

where dots stand for different distributions of the three covariant derivatives such that \mathcal{O}_3 is an eigenstate of the dilatation operator. The precise identification of the string state dual to this operator requires detailed analysis which is beyond the scope of this work. Assuming that it corresponds to the $n = 3$ spiky string we then find from (1.9) the following prediction for the strong-coupling expansion of its anomalous dimension

$$E_{\text{spiky}}(S = J = n = 3) = 2\sqrt{2}\sqrt[4]{\lambda} \left[1 + \frac{41}{24} \frac{1}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right] . \quad (1.19)$$

⁴Note that the harmonic sum can be written in terms of the logarithmic derivative of the Gamma function, $\psi(z) = \frac{d}{dz} \log \Gamma(z)$, as follows: $H_r = \psi(r+1) + \gamma_E$.

⁵The value $n = 1$ is special as this is the minimal value corresponding to the ground state. This can be an explanation of the success of the matching in the case of the Konishi operator.

For comparison, $E_{\text{folded}}(S = J = 3, m = 1) = \sqrt{6} \sqrt[4]{\lambda} \left[1 + \frac{13}{8} \frac{1}{\sqrt{\lambda}} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right]$ is lower.

Let us mention also the limit of large number of spikes $n \rightarrow \infty$. In the “long” string case it is possible to scale E, S, J with n keeping $\frac{E+S}{n^2}$, $\frac{E-S}{n}$, and $\frac{J}{n}$ fixed [20]. In this limit the spiky string approaches the boundary of AdS_5 ; the corresponding solution can be interpreted as describing a periodic-spike string moving in $AdS_3 - \text{pp-wave} \times S^1$ background [20, 33]. In the “short” string case we are interested here we may take n large while keeping S, J fixed. We then find that while the classical contribution in (1.8) has a finite limit, the one-loop correction goes as $\log n$ ⁶,

$$E \stackrel{n \rightarrow \infty}{\simeq} 2 \sqrt{\sqrt{\lambda} S} \left[1 + \underbrace{\frac{1}{2\sqrt{\lambda}} \left(S + \frac{J^2}{4S} \right)}_{\text{classical}} + \underbrace{\log \frac{n e^{\gamma_E}}{2} - \frac{1}{4} + \mathcal{O}\left(\frac{1}{n}\right)}_{\text{quantum}} + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right]. \quad (1.20)$$

As was mentioned above, the symmetric spiky string and the m -folded string are special in the sense that the corresponding algebraic curve reduces to a curve with two cuts on the real axis on opposite sides of the origin. Such a solution is fully characterized by the positive mode numbers $-n_L$ and n_R associated with the left and right cuts. These mode numbers appear in the corresponding flat-space solution that generalizes both (1.6) and (1.7). The symmetric spiky string solution has $(n_L, n_R) = (1, n - 1)$ while the m -folded string has $(n_L, n_R) = (m, m)$. The (n_L, n_R) solution in $AdS_5 \times S^5$ describes a generalised symmetric spiky string with a possible AdS_3 winding discussed in [18] (see also Section (5.1.3) in [23]). The corresponding expression for the one-loop corrected energy is the following generalization of both (1.8),(1.9) and (1.10),(1.11)

$$E_{(n_L, n_R)} = \sqrt{\frac{4n_L n_R}{n_L + n_R}} \sqrt{\lambda} S \left[1 + \frac{1}{\sqrt{\lambda}} (a_{01}(n_L, n_R) + \dots) + \dots \right], \quad (1.21)$$

$$E_{(n_L, n_R)}^2 = \frac{4n_L n_R}{n_L + n_R} \sqrt{\lambda} S \left[1 + \frac{2}{\sqrt{\lambda}} a_{01}(n_L, n_R) + \mathcal{O}\left(\frac{1}{(\sqrt{\lambda})^2}\right) \right] + J^2 + \gamma S^2 + \dots \quad (1.22)$$

$$a_{01}(n_L, n_R) = \frac{1}{2} [q(n_L) + q(n_R)], \quad (1.23)$$

where q was defined in (1.15). Indeed, as one can readily see (cf. (1.13),(1.14),(1.17))

$$a_{01}^{\text{spiky}}(n) = a_{01}(1, n - 1), \quad a_{01}^{\text{folded}}(m) = a_{01}(m, m). \quad (1.24)$$

The “additive” structure of a_{01} in (1.23) implies that to this low order of the “short-string”(or “near-flat-space”) expansion the contributions of the “left” and “right” modes simply add up, which should be a consequence of the integrability.

The structure of the rest of this paper is as follows. We shall start in section 2 with a review of the algebraic curve description of the spiky string. In section 3 we shall compute the general expression for the 1-loop correction to its energy using the algebraic curve approach. In section 4 we shall consider the “short” string (small spin) expansion of this

⁶That may be implying that such a limit is not well-defined in the strong-coupling expansion.

1-loop correction finding the 1-loop coefficient in (1.13). In section 5 we shall repeat the same analysis for the m -folded string leading to (1.14) and present an explanation for the close relation between the two expressions in (1.13) and in (1.14) implied by (1.23). Appendices A and C contain some technical details and in Appendix B we present for completeness the expressions for the 1-loop energy of the spiky and m -folded strings in the opposite “long-string” (large spin) limit.

2 Algebraic curve description of the symmetric spiky string

As we have mentioned above, starting with the Bethe Ansatz equations, the large-spin symmetric spiky string can be described by taking a thermodynamic limit, which leads to a 2-cut solution. The filling fractions associated with the two cuts can be traded for the two constant mode numbers along the cuts (see, e.g., [34]). These mode numbers can be identified by analyzing the relevant string solution in flat space which is a combination of right and left moving modes. For instance, the folded string with $m = 1$ has the same number of right and left moving excitations, i.e. $n_L = n_R$. This suggests that in this case the two cuts should be symmetric with modes ± 1 [34]. Following the same logic, in the spiky string case [19] one should get the two asymmetric cuts with mode numbers $n_L = 1$ and $n_R = n - 1$, where n is the number of spikes. This input data was used in the discussion of the large S spiky strings in the context of the all-loop Bethe Ansatz equations [24]. The strong coupling limit of the Bethe Ansatz equations [25] is the starting point for the construction of the algebraic curve needed for the semiclassical expansion [7].

According to [25], after a suitable rescaling, these equations reduce to⁷

$$\oint dx' \rho(x') \frac{1 - \frac{1}{xx'}}{(x - x')(1 - \frac{1}{x'^2})} = \pi n(x) \left(1 - \frac{1}{x^2}\right) - \frac{2\pi}{x} \mathcal{J}. \quad (2.1)$$

Here the density $\rho(x)$ describes the momentum carrying roots (there are $S \rightarrow \infty$ of them) and is supported on a certain union of real cuts. The mode number function $n(x)$ is piecewise constant on the cuts. It is convenient to define

$$\tilde{\rho}(x) = \frac{x^2}{x^2 - 1} \rho(x), \quad \tilde{n}(x) = n(x) - \frac{2x}{x^2 - 1} \mathcal{J}. \quad (2.2)$$

Then (2.1) becomes

$$\oint dx' \frac{\tilde{\rho}(x')}{x - x'} \left(1 - \frac{1}{xx'}\right) = \pi \tilde{n}(x) \left(1 - \frac{1}{x^2}\right). \quad (2.3)$$

The zero momentum and normalization conditions are

$$\oint dx \frac{\tilde{\rho}(x)}{x} = 0, \quad \oint dx \tilde{\rho}(x) \left(1 - \frac{1}{x^2}\right) = 4\pi S. \quad (2.4)$$

Finally, the energy can be expressed as

$$\mathcal{E} - S - \mathcal{J} = \frac{1}{2\pi} \oint dx \frac{\tilde{\rho}(x)}{x^2}. \quad (2.5)$$

⁷Below we use the notation $\mathcal{E} = E/\sqrt{\lambda}$, $S = S/\sqrt{\lambda}$, $\mathcal{J} = J/\sqrt{\lambda}$.

2.1 Solution of the integral equation in terms of a resolvent

Let us focus on the 2-cut case

$$C_1 = (d, c), \quad C_2 = (b, a), \quad d < c < -1, \quad 1 < b < a, \quad (2.6)$$

and define the function⁸

$$f(z) = \sqrt{z-a} \sqrt{z-b} \sqrt{z-c} \sqrt{z-d}. \quad (2.7)$$

The resolvent $G(z)$ is defined as

$$G(z) = \frac{1}{\pi} \int_{C_1 \cup C_2} dx' \frac{f(z)}{f(x' + i\varepsilon)} \frac{\tilde{n}(x')}{x' - z}. \quad (2.8)$$

By standard manipulations, one can show that the resolvent obeys

$$G(x \pm i0) = \pm \tilde{\rho}(x) + i \tilde{n}(x), \quad x \in C_1 \cup C_2, \quad (2.9)$$

with the explicit expression for $\tilde{\rho}(x)$

$$\tilde{\rho}(x) = \frac{1}{\pi} \oint dx' \operatorname{sign}(xx') \left| \frac{f(x)}{f(x')} \right| \frac{\tilde{n}(x')}{x' - x}. \quad (2.10)$$

The main result is that $\tilde{\rho}(x)$ satisfies (2.3) provided

$$G(0) = G_0 = G_1 = 0, \quad (2.11)$$

where the constants G_0, G_1 are extracted from

$$G(z) \stackrel{z \rightarrow 0}{\sim} G(0) + H_1 z + \dots, \quad (2.12)$$

$$G(z) \stackrel{z \rightarrow \infty}{\sim} G_0 z + G_1 + G_2 \frac{1}{z} + \dots, \quad (2.13)$$

The zero momentum condition in (2.4) is then automatically satisfied and the charges in (2.4), (2.5) are given by

$$\mathcal{S} = -\frac{i}{4}(G_2 + H_1), \quad \mathcal{E} - \mathcal{S} - \mathcal{J} = \frac{i}{2} H_1. \quad (2.14)$$

2.2 Special case of the spiky string

For the string with n symmetric spikes, we are to choose the mode number function as

$$n(x) = \begin{cases} -1, & d < x < c < -1, \\ n-1, & 1 < b < x < a, \end{cases} \quad (2.15)$$

and then the resolvent turns out to be

$$G(z) = -\frac{2i n}{\pi \sqrt{(a-c)(b-d)}} \frac{f(z)}{(b-z)(z-c)} \left[(b-c) \Pi \left(\frac{(a-b)(z-c)}{(a-c)(z-b)}, r \right) + (z-b) \mathbb{K}(r) \right] \\ - i - 2i \mathcal{J} \left[\frac{z}{z^2-1} + \frac{f(z)}{2} \left(\frac{1}{f(1)(1-z)} - \frac{1}{f(-1)(1+z)} \right) \right], \quad (2.16)$$

where $r \equiv \frac{(a-b)(c-d)}{(a-c)(b-d)}$. The constants appearing in the asymptotic expansion of the resolvent at $z = 0$ and $z = \infty$ are collected in Appendix A.

⁸We choose the standard cut for the square roots.

3 Semi-classical quantization of the spiky string in the algebraic curve approach

A review of the algebraic curve approach can be found, e.g., in [35] whose notation we shall follow.

3.1 Classical data

The algebraic curve is characterized by a set of quasi momenta $\widehat{p}_i, \widetilde{p}_i$, $i = 1, 2, 3, 4$. They enter the monodromy matrix of the Lax connection for the integrable dynamics of the classical $AdS_5 \times S^5$ superstring that has the eigenvalues

$$\{e^{i\widehat{p}_1}, e^{i\widehat{p}_2}, e^{i\widehat{p}_3}, e^{i\widehat{p}_4} | e^{i\widetilde{p}_1}, e^{i\widetilde{p}_2}, e^{i\widetilde{p}_3}, e^{i\widetilde{p}_4}\}. \quad (3.1)$$

These eigenvalues are roots of the characteristic polynomial and define an 8-sheeted Riemann surface. The classical algebraic curve has macroscopic cuts connecting various pairs of sheets. Around each cut, we have

$$p_i^+ - p_j^- = 2\pi n_{ij}, \quad x \in C_n^{ij}, \quad (3.2)$$

where n_{ij} is an integer associated with a given cut. The possible combinations of sheets (a.k.a. polarizations) that are relevant for $AdS_5 \times S^5$ are

$$i = \widetilde{1}, \widetilde{2}, \widehat{1}, \widehat{2}, \quad j = \widetilde{3}, \widetilde{4}, \widehat{3}, \widehat{4}. \quad (3.3)$$

The general properties of quasi-momenta are reviewed in [35]. In our case, similar to [7], the non-trivial quasi-momentum is \widehat{p}_2 with all other being determined by the cut structure, inversion relations, and Virasoro constraints, precisely as in [7]. In terms of the resolvent, it can be shown that the required relation is

$$p_{\widehat{2}}(x) = \pi \left(-i G(x) + 2 \mathcal{J} \frac{x}{x^2 - 1} \right). \quad (3.4)$$

3.2 Off-shell fluctuations

To compute the one-loop correction to string energy in the algebraic curve approach one is first to find normal mode frequencies for fluctuations around the classical solution. One is then to quantize the associated set of effective oscillators and evaluate the energy correction by simply summing up zero-point energies, taking into account the Fermi-Bose statistics.

The key ingredient are the so-called off-shell fluctuation energies $\Omega^{ij}(x)$, where i, j label two sheets of the algebraic curve. These are functions of the spectral parameter x which are actually simpler than the normal mode physical frequencies. Off-shell fluctuation energies reduce to them at special non-trivial points x_k^{ij} associated with the i -th and j -th sheets that satisfy

$$p_i(x_k^{ij}) - p_j(x_k^{ij}) = 2\pi k. \quad (3.5)$$

Here, k can be regarded as a mode number for the k -th normal mode frequency with polarization (i, j) , i.e. $\Omega^{ij}(x_k^{ij})$.

Due to the symmetries of the classical solution, the 8+8 bosonic and fermionic frequencies can be written in terms of only two independent off-shell fluctuations [30]. The result in our case is

$$\Omega_S(x) \equiv \Omega^{\widetilde{2}\widetilde{3}}(x) = \Omega^{\widetilde{2}\widetilde{4}}(x) = \Omega^{\widetilde{1}\widetilde{3}}(x) = \Omega^{\widetilde{1}\widetilde{4}}(x), \quad (3.6)$$

$$\Omega_A(x) \equiv \Omega^{\widehat{2}\widehat{3}}(x). \quad (3.7)$$

All other frequencies are given by the following expressions

$$\begin{aligned} \Omega^{(1)}(x) &= \Omega^{\widehat{1}\widehat{4}}(x) = -\Omega_A\left(\frac{1}{x}\right) - 2, \\ \Omega^{(2)}(x) &= \Omega^{\widehat{1}\widehat{3}}(x) = \Omega^{\widehat{2}\widehat{4}}(x) = \frac{1}{2}\Omega_A(x) - \frac{1}{2}\Omega_A\left(\frac{1}{x}\right) - 1, \\ \Omega^{(3)}(x) &= \Omega^{\widehat{2}\widetilde{3}}(x) = \Omega^{\widehat{2}\widetilde{4}}(x) = \Omega^{\widetilde{3}\widetilde{1}}(x) = \Omega^{\widetilde{3}\widetilde{2}}(x) = \frac{1}{2}\Omega_A(x) + \frac{1}{2}\Omega_S(x), \\ \Omega^{(4)}(x) &= \Omega^{\widehat{1}\widetilde{3}}(x) = \Omega^{\widehat{1}\widetilde{4}}(x) = \Omega^{\widehat{4}\widetilde{1}}(x) = \Omega^{\widehat{4}\widetilde{2}}(x) = \frac{1}{2}\Omega_S(x) - \frac{1}{2}\Omega_A\left(\frac{1}{x}\right) - 1. \end{aligned} \quad (3.8)$$

Thus, we just need to compute the two frequencies $\Omega_{A,S}(x)$ as in the folded string case. A simple calculation gives

$$\Omega_A(x) = -\frac{2}{1+f(0)} \left(1 - \frac{f(x)}{x^2-1}\right), \quad (3.9)$$

$$\Omega_S(x) = \frac{1}{1+f(0)} \frac{f(1)(x+1) - f(-1)(x-1)}{x^2-1}, \quad (3.10)$$

where $f(x)$ has been defined in (2.7).

3.3 One-loop correction to the energy

The one-loop energy is given, in general, by

$$\delta E^{1-\text{loop}} = \frac{1}{2} \sum_{ij} (-1)^{F_{ij}} \oint \frac{dx}{2\pi i} \Omega^{ij} G_{ij}, \quad (3.11)$$

where

$$G_{ij} = \partial_x \log \sin \frac{p_i - p_j}{2}. \quad (3.12)$$

In our case

$$\begin{aligned} \sum_{ij} (-1)^{F_{ij}} \Omega^{ij} G_{ij} &= 4\Omega_S G_{\widetilde{2}\widetilde{3}} + \Omega_A G_{\widehat{2}\widehat{3}} + \Omega^{(1)} G_{\widehat{1}\widehat{4}} + 2\Omega^{(2)} G_{\widehat{1}\widehat{3}} \\ &\quad - 4\Omega^{(3)} G_{\widehat{2}\widetilde{4}} - 4\Omega^{(4)} G_{\widetilde{2}\widehat{4}}. \end{aligned} \quad (3.13)$$

As in the case of the folded string [7], the one-loop correction can be written as

$$\delta E^{1-\text{loop}} = \delta E^{(1)} + \delta E^{(2)} + \delta E^{(3)}, \quad (3.14)$$

where

$$\begin{aligned}
\delta E^{(1)} &= \int_{-1}^1 \frac{dz}{\pi} \operatorname{Im}(p_{\widehat{2}} - p_{\widehat{2}}) \partial_z \operatorname{Im}(\Omega_S - \Omega_A), \\
\delta E^{(2)} &= \int_{-1}^1 \frac{dz}{\pi} \operatorname{Im} \left[\partial_z \Omega_S \log \frac{(1 - e^{-i p_{\widehat{2}} + i \bar{p}_{\widehat{2}}})(1 - e^{-i p_{\widehat{2}} - i \bar{p}_{\widehat{2}}})}{(1 - e^{-2i p_{\widehat{2}}})^2} \right. \\
&\quad \left. - \partial_z \Omega_A \log \frac{(1 - e^{-2i p_{\widehat{2}}})(1 - e^{-i p_{\widehat{2}} + i \bar{p}_{\widehat{2}}})}{(1 - e^{-i p_{\widehat{2}} - i \bar{p}_{\widehat{2}}})^2} \right], \\
\delta E^{(3)} &= \frac{2}{1 + f(0)} \int_{(d,c) \cup (b,a)} \frac{dx}{2\pi i} \frac{f(x)}{x^2 - 1} \partial_x \log \sin p_{\widehat{2}}.
\end{aligned} \tag{3.15}$$

In the first two integrals we have defined

$$x = x(z) = z + \sqrt{z^2 - 1}, \quad \bar{p}_{\widehat{2}} = p_{\widehat{2}} \left(\frac{1}{x(z)} \right). \tag{3.16}$$

The third integral is done above the cuts $(d, c) \cup (b, a)$.

4 One-loop correction to the energy of the short spiky string

The above general expressions allow one to find the one-loop correction to the spiky string energy. In particular, in the “*long string*” (large spin) limit one recovers the results found in [24], as we summarize in Appendix B. Here we shall concentrate on the opposite “*short string*” limit.

4.1 Expansion of the classical data

We define the short string limit as

$$\mathcal{S} \equiv \frac{1}{2} s^2 \rightarrow 0, \quad \text{with} \quad \mathcal{J} = \rho \mathcal{S} \rightarrow 0, \quad \rho \equiv \frac{J}{S} = \text{fixed}. \tag{4.1}$$

Then the endpoints of the two cuts have the following expansion

$$a = 1 + \frac{2\sqrt{2}}{\sqrt{n(n-1)}} s + \frac{4}{n(n-1)} s^2 + \frac{n(\rho^2 n + 4) + 20}{8\sqrt{2}(n(n-1))^{3/2}} s^3 + \mathcal{O}(s^4), \tag{4.2}$$

$$b = 1 + \frac{1}{8} \sqrt{\frac{n}{2(n-1)^3}} \rho^2 s^3 - \frac{n(n\rho^2 - 12) + 20}{256\sqrt{2}n(n-1)^{5/2}} \rho^2 s^5 + \mathcal{O}(s^6), \tag{4.3}$$

$$c = -1 - \frac{n\rho^2}{8\sqrt{2}n(n-1)} s^3 + \frac{n(n(\rho^2 + 8) - 28) + 20}{256\sqrt{2}n(n-1)^{3/2}} \rho^2 s^5 + \mathcal{O}(s^6), \tag{4.4}$$

$$d = -1 - 2\sqrt{\frac{2(n-1)}{n}} s + \left(\frac{4}{n} - 4 \right) s^2 - \frac{(n(n(\rho^2 + 24) - 44) + 20)s^3}{8n\sqrt{2}n(n-1)} + \mathcal{O}(s^4). \tag{4.5}$$

The classical energy is then given by

$$E_0 = \sqrt{\lambda} \left[\sqrt{\frac{2(n-1)}{n}} s + \frac{(n[n(\rho^2 + 4) - 10] + 10)s^3}{8n\sqrt{2}(n-1)n} + \mathcal{O}(s^4) \right]. \tag{4.6}$$

which can be recognized as the classical part in (1.8).

4.2 The one-loop coefficient $a_{01}(n)$

The small s expansion of the one-loop correction starts with

$$E_{1\text{-loop}} = \sqrt{\frac{2(n-1)}{n}} a_{01}^{\text{spiky}} s + \dots \quad (4.7)$$

The calculation of $a_{01}(n)$ can be done along the lines explained in [7]. The trick is to evaluate the one-loop integrals in (3.14) splitting the integration region into three parts ($\Lambda_{1,2}$ are auxiliary parameters)

$$z \in (0, 1) = (0, 1 - s \Lambda_1) \cup (1 - s \Lambda_1, 1 - s^3 \Lambda_2) \cup (1 - s^3 \Lambda_2, 1). \quad (4.8)$$

The integrals can be computed in each interval by a rather straightforward expansion in $s \rightarrow 0$. Finally, the three divergent results are merged and the cutoffs $\Lambda_{1,2}$ can be sent to infinity. This procedure is known as *matched asymptotic expansion* [36] (see Appendix C for a simple example). The computation is rather complicated and finally leads to the following results

$$a_{01}^{\text{spiky}}(n) \Big| \begin{array}{cccccccccc} n & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \dots \\ \hline & -\frac{1}{4} & \frac{7}{48} & \frac{43}{120} & \frac{1699}{3360} & \frac{3119}{5040} & \frac{9853}{13860} & \frac{568291}{720720} & \frac{2466643}{2882880} & \dots \end{array}$$

The closed expression for this sequence is

$$a_{01}^{\text{spiky}}(n) = -\frac{1}{8} + \frac{1}{2} q(n-1), \quad (4.9)$$

where

$$q(r) = -\frac{3}{4r} + 2H_r - H_{2r}, \quad H_r = \sum_{\ell=1}^r \frac{1}{\ell}. \quad (4.10)$$

The analytic expression (4.9) was carefully checked by a numerical evaluation of the one-loop correction extrapolated to $s \rightarrow 0$: we verified the values of a_{01}^{spiky} and its independence of $\rho = J/S$ with 10^{-8} accuracy.

5 m -folded string and relation to the spiky string case

Let us now consider the folded string case. We will not repeat all the details since they are completely similar to the spiky string case. The spinning string solution with m folds is described by a 2-cut solution with mode numbers $\pm m$ [18]. The classical energy reads

$$E_0 = m \sqrt{\lambda} s \left(1 + \frac{3 + 2\rho^2}{16} s^2 + \dots \right), \quad (5.1)$$

$$s = \sqrt{\frac{2S}{m\sqrt{\lambda}}}, \quad \rho = \frac{J}{S}. \quad (5.2)$$

The one-loop correction turns out to be linear in $s \rightarrow 0$

$$E_{1\text{-loop}} = m a_{01}^{\text{folded}}(m) s + \dots, \quad (5.3)$$

leading to the expression in Eq. (1.10). Again, the values of $a_{01}^{\text{folded}}(m)$ can be determined analytically⁹

$$a_{01}^{\text{folded}}(m) \begin{array}{c|cccccccccc} m & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots \\ \hline & -\frac{1}{4} & \frac{13}{24} & \frac{29}{30} & \frac{2119}{1680} & \frac{3749}{2520} & \frac{23171}{13860} & \frac{658381}{360360} & \frac{2827003}{1441440} & \frac{25478473}{12252240} & \frac{508697569}{232792560} & \dots \end{array} \quad (5.4)$$

and one can find a closed formula for this sequence¹⁰, namely,

$$a_{01}^{\text{folded}}(m) = -\frac{3}{4m} + 2H_m - H_{2m} = q(m). \quad (5.5)$$

This is the same function q as the one that appeared in the spiky string case (4.9). A possible explanation of the close relation between the two 1-loop coefficients is a conjectured decoupling of the left- and right- moving string modes in the short-string limit.

Indeed, from the algebraic curve point of view the m -folded string has two symmetric cuts, one with mode number $-m$ and the other with mode number $+m$. On the other hand, the spiky string with n spikes has two asymmetric cuts, one with mode number -1 and the other with mode number $n-1$. We may assume that, in the short string limit, the contributions from the left and right cuts simply add. However, before combining them, we have to take into account the different normalizations of the cut endpoints. Indeed, the short string expansion of the largest endpoint a of the positive cut reads

$$a^{\text{folded}} = 1 + 2s + 2s^2 + \dots, \quad (5.6)$$

$$a^{\text{spiky}} = 1 + 2\tilde{s} + 2\tilde{s}^2 + \dots, \quad \tilde{s} = \frac{\sqrt{2}}{\sqrt{n(n-1)}} s. \quad (5.7)$$

Taking into account the ratio \tilde{s}/s and the normalization of a_{01}^{spiky} in (4.7) this means that in the spiky string case the sum of the contribution of the cut with the mode number -1 (half of folded string with $m=1$) and of the cut with the mode number $n-1$ (half of a folded string with $m=n-1$) reads

$$a_{01}^{\text{spiky}}(n) = \frac{1}{2} \times a_{01}^{\text{folded}}(m=1) + \frac{1}{2} \times a_{01}^{\text{folded}}(m=n-1). \quad (5.8)$$

This is exactly the relation (4.9) found in the spiky string case.

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⁹The first entries of the list have been independently confirmed in the recent paper [37].

¹⁰This requires analyzing many more values than shown in the above table.

A Expansion of the spiky string resolvent at $z \rightarrow 0$ and $z \rightarrow \infty$

The expansion of $G(z)$ in (2.16) at $z \rightarrow 0$ and $z \rightarrow \infty$ determines the constants $G(0)$, G_0 , G_1 , G_2 , H_1 . They were computed in [25]:

$$G(0) = -i + f(0) \left[\frac{2in}{\pi} \frac{b-c}{bc\sqrt{(a-c)(b-d)}} \bar{\Pi} + \frac{i\mathcal{J}}{c} \left(\frac{1-c}{f(1)} + \frac{1+c}{f(-1)} \right) \right], \quad (\text{A.1})$$

$$G_0 = \frac{2in}{\pi\sqrt{(a-c)(b-d)}} \mathbb{K} + i\mathcal{J} \left(\frac{1}{f(1)} + \frac{1}{f(-1)} \right), \quad (\text{A.2})$$

$$G_1 = -i + \frac{2in}{\pi} \frac{b-c}{\sqrt{(a-c)(b-d)}} \Pi + \frac{2in}{\pi} \frac{c}{\sqrt{(a-c)(b-d)}} \mathbb{K} + i\mathcal{J} \left(\frac{1}{f(1)} - \frac{1}{f(-1)} \right), \quad (\text{A.3})$$

$$G_2 = -2i\mathcal{J} + \frac{2in}{\pi} \frac{b^2 - c^2}{\sqrt{(a-c)(b-d)}} \Pi + \frac{2in}{\pi} \frac{(b-c)^2(a-b)}{(a-c)\sqrt{(a-c)(b-d)}} \Pi' + \frac{2in}{\pi} \frac{c^2}{\sqrt{(a-c)(b-d)}} \mathbb{K}(r) + i\mathcal{J} \left(\frac{1}{f(1)} + \frac{1}{f(-1)} \right) - \frac{i}{2}(a+b+c+d), \quad (\text{A.4})$$

$$H_1 = 2i\mathcal{J} + \frac{i}{2} \left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a} - \frac{1}{d} \right) - \frac{2in}{\pi} \frac{(b-c)^2(a-b)}{b^3c(a-c)\sqrt{(a-c)(b-d)}} \bar{\Pi}' f(0) + \frac{i\mathcal{J}}{bc} f(0) \left(\frac{(1-c)(1-b)}{f(1)} + \frac{(1+c)(1+b)}{f(-1)} \right). \quad (\text{A.5})$$

The coefficients here are defined in terms of the elliptic functions as follows

$$\mathbb{K} = \mathbb{K}(r), \quad \Pi = \Pi(v, r), \quad \bar{\Pi} = \Pi\left(\frac{c}{b}v, r\right), \quad (\text{A.6})$$

$$v \equiv \frac{a-b}{a-c}, \quad r \equiv \frac{(a-b)(c-d)}{(a-c)(b-d)}, \quad (\text{A.7})$$

$$\Pi'(v, r) = \frac{\partial \Pi(v, r)}{\partial v} = \frac{1}{2(r-v)(v-1)} \left[\mathbb{E}(r) + \frac{r-v}{v} \mathbb{K}(r) + \frac{v^2-r}{v} \Pi(v, r) \right] \quad (\text{A.8})$$

B Large spin limit of the spiky string and m -folded string

The large spin limit of the spiky and m -folded string can be computed in a very simple way from our expressions for the algebraic curve and adapting the calculation performed in [7] for the standard folded string with $m = 1$. In the notation of that paper, the one-loop correction is computed in terms of the three contributions $\delta E^{(1)}$, $\delta E^{(2)}$, and $\delta E^{(3)}$ which are the symmetric cases of Eqs. (3.15). Also, we can further split $\delta E^{(3)}$ into an “anomaly”

contribution plus a remainder term, i.e. $\delta E^{(3)} = \delta E_{an}^{(3)} + \delta E_m^{(3)}$. The leading and next-to-leading contributions come only from $\delta E^{(1)}$ and $\delta E_m^{(3)}$. The other terms are suppressed as $\mathcal{O}(1/\log S)$.

The long string limit $S \rightarrow \infty$ is achieved when the cut (b, a) endpoints have the asymptotic behaviour $a \rightarrow \infty$ and $b \rightarrow 1$ with

$$S = \frac{1}{2\pi} a, \quad \mathcal{J} = \frac{1}{\pi} \sqrt{b^2 - 1} \log \frac{a}{b}. \quad (\text{B.1})$$

The one-loop correction to the energy of the “long” 1-folded string turns out to be

$$E_{1\text{-loop}} = -\frac{3 \log 2}{\pi} \log \bar{S} + \frac{6 \log 2}{\pi} + 1 + \mathcal{O}\left(\frac{1}{\log \bar{S}}\right), \quad S \equiv 8\pi \bar{S}. \quad (\text{B.2})$$

The derivation of this result involves fixing the ratio $\ell = J/\log S$ and sending $\ell \rightarrow 0$ in the end (the only important feature is the relation between J and $\log S$ coming from (B.1)).

In the spiky string case, the large spin limit is obtained by considering the two cuts $(d, c) \cup (b, a)$ in the limit

$$d = -u a, \quad a \rightarrow +\infty, \quad (\text{B.3})$$

where u is a real positive constant. The conditions (2.11) on the resolvent $G_0 = G_1 = 0$ give important information in this limit. We start with G_0 that admits the large a expansion

$$G_0 = \frac{G_{0,-1}}{a} + \mathcal{O}\left(\frac{1}{a^2}\right). \quad (\text{B.4})$$

The vanishing of $G_{0,-1}$ gives the basic relation between \mathcal{J} and $\log a$

$$\mathcal{J} \left(\frac{1}{\sqrt{(b-1)(1-c)}} + \frac{1}{\sqrt{-(b+1)(c+1)}} \right) = \frac{n}{\pi} \log \frac{16 a u}{(b-c)(u+1)}. \quad (\text{B.5})$$

Expanding G_1 we find instead two non trivial terms

$$G_1 = G_{1,1} a + G_{1,0} + \mathcal{O}\left(\frac{1}{a}\right). \quad (\text{B.6})$$

The vanishing of $G_{1,1}$ gives

$$u = \cot^2 \frac{\pi}{2n}. \quad (\text{B.7})$$

The vanishing of $G_{1,0}$ gives

$$\frac{b+c+2}{\sqrt{-(b+1)(c+1)}} + \frac{b+c-2}{\sqrt{(b-1)(1-c)}} = 0 \quad \longrightarrow \quad b+c=0, \quad (\text{B.8})$$

and the two cuts become symmetric in this limit. Using these results to simplify the expression for the spin, we easily obtain the following leading-order expression

$$S = \frac{n a \sqrt{u}}{4\pi} + \mathcal{O}(1). \quad (\text{B.9})$$

If we trade a for \mathcal{S} and use $b + c = 0$, we can write the relation between \mathcal{J} and $\log \mathcal{S}$ in the following form (which is a required modification of (B.1))

$$\mathcal{J} = \frac{n}{2\pi} \sqrt{b^2 - 1} \log \left(\frac{2}{n} \sin \frac{\pi}{n} \frac{\overline{\mathcal{S}}}{b} \right). \quad (\text{B.10})$$

From this equation, it is a straightforward exercise to obtain the following $n > 2$ generalization of (B.2)

$$E_{1-\text{loop}}^{\text{spiky}} = \frac{n}{2} \left[-\frac{3 \log 2}{\pi} \log \left(\frac{2}{n} \sin \frac{\pi}{n} \overline{\mathcal{S}} \right) + \frac{6 \log 2}{\pi} + 1 \right] + \mathcal{O} \left(\frac{1}{\log \overline{\mathcal{S}}} \right), \quad (\text{B.11})$$

which is in agreement with the Bethe Ansatz calculation of [24]. This agreement is, of course, expected due to the general results of [29]. A completely similar treatment for the m -folded string case gives another $m > 1$ generalization of (B.2)

$$E_{1-\text{loop}}^{\text{folded}} = m \left(-\frac{3 \log 2}{\pi} \log \frac{\overline{\mathcal{S}}}{m} + \frac{6 \log 2}{\pi} + 1 \right) + \mathcal{O} \left(\frac{1}{\log \overline{\mathcal{S}}} \right). \quad (\text{B.12})$$

C On the asymptotic evaluation of 2-scale integrals

Let us consider the integral

$$I(s) = \int_0^1 dx f(x), \quad f(x) = \frac{1}{(x-1-s)(x-1-s^3)}. \quad (\text{C.1})$$

It can be computed exactly and expanded for $s \rightarrow 0$

$$I(s) = \frac{1}{s(s^2-1)} \log \frac{s^2}{s^2-s+1} = -2 \frac{\log s}{s} - 1 + \left(\frac{1}{2} - 2 \log s \right) s - \frac{s^2}{3} + \mathcal{O}(s^3). \quad (\text{C.2})$$

Let us now describe the general strategy of how to obtain this asymptotic expansion without requiring the knowledge of the exact integral. We first split the integral into three parts as

$$I(s) = \left(\int_0^{1-s\Lambda_1} + \int_{1-s\Lambda_1}^{1-s^3\Lambda_2} + \int_{1-s^3\Lambda_2}^1 \right) dx f(x) \quad (\text{C.3})$$

This can be written as

$$I(s) = \int_0^{1-s\Lambda_1} dx f(x) - s \int_{\Lambda_1}^{s^2\Lambda_2} d\tau f(1-s\tau) - s^3 \int_{\Lambda_2}^0 d\xi f(1-s^3\xi). \quad (\text{C.4})$$

Then we directly expand each integrand in powers of s

$$f(x) = \frac{1}{(x-1)^2} + \frac{s}{(x-1)^3} + \frac{s^2}{(x-1)^4} + \mathcal{O}(s^3), \quad (\text{C.5})$$

$$-s f(1-s\tau) = -\frac{1}{\tau(\tau+1)s} + \frac{s}{\tau^2(\tau+1)} + \mathcal{O}(s^3), \quad (\text{C.6})$$

$$-s^3 f(1-s^3\xi) = -\frac{1}{(\xi+1)s} + \frac{\xi s}{\xi+1} + \mathcal{O}(s^3). \quad (\text{C.7})$$

Performing the integrals term by term we find

$$\int_0^{1-s\Lambda_1} dx f(x) = \frac{2 - 3\Lambda_1 + 6\Lambda_1^2}{6\Lambda_1^3} \frac{1}{s} - 1 + \frac{s}{2} - \frac{s^2}{3} + \mathcal{O}(s^3), \quad (\text{C.8})$$

$$\begin{aligned} -s \int_{\Lambda_1}^{s^2\Lambda_2} d\tau f(1-s\tau) &= \left(-2\log s + \log \frac{\Lambda_1}{\Lambda_1+1} - \log \Lambda_2 - \frac{1}{\Lambda_2} \right) \frac{1}{s} \\ &\quad + \left(-2\log s + \log \frac{\Lambda_1}{\Lambda_1+1} - \log \Lambda_2 + \frac{1}{\Lambda_1} + \Lambda_2 \right) s + \mathcal{O}(s^3), \end{aligned} \quad (\text{C.9})$$

$$-s^3 \int_{\Lambda_2}^0 d\xi f(1-s^3\xi) = \log(\Lambda_2+1) \frac{1}{s} + (\log(\Lambda_2+1) - \Lambda_2) s + \mathcal{O}(s^3). \quad (\text{C.10})$$

Summing up and sending $\Lambda_{1,2} \rightarrow \infty$, all the divergences cancel and we recover precisely the expansion in (C.2).

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